ACTIVITY FIVE: The Pythagorean Identity in Trigonometry

Let ABC be a triangle with right angle at C. Its legs have lengths a and b and the length of the hypotenuse is c.

Apply the scaling factor $\frac{1}{c}$ to create a similar triangle $A'B'C'$ with legs $\frac{a}{c}$ and $\frac{b}{c}$ and hypotenuse 1.

When the Pythagorean Theorem is used on the new triangle, we have

$$\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1^2 = 1$$

In right triangle trigonometry, we define the sine and cosine functions as follows. Consider an acute angle θ in triangle ABC. The sine of θ is the ratio created by taking the length of the opposite side and dividing by the length of the hypotenuse. In our triangle, I've arbitrarily placed θ at vertex A, and so

$$\sin(\theta) = \frac{a}{c}$$

Because ABC and $A'B'C'$ are similar, the corresponding angles are congruent. Thus, we may also write:

$$\sin(\theta) = \frac{\frac{a}{c}}{1} = \frac{a}{c}$$

Note that the sine is dependent only on the angle and not the size of the triangle.

Similarly, the cosine of θ is the ratio created by taking the length of the adjacent side and dividing by the length of the hypotenuse: $\cos(\theta) = \frac{b}{c}$. (To confirm the independence of the angle, consider $A'B'C'$ where $\cos(\theta) = \frac{\frac{b}{c}}{1} = \frac{b}{c}$.)

Now let's go back to $\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1$ and "substitute equals for equals" to arrive at the Pythagorean Identity for Trigonometry.

$$(\sin(\theta))^2 + (\cos(\theta))^2 = 1$$