Perspective Drawing and Projective Geometry

- Albrecht Dürer (1471–1528), Leonardo Da Vinci (1452–1519), Brook Taylor (1685–1731)
- Industrial Revolution
- Properties shared by 2 perspective views of same scene?
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Evaluate the following arguments:

a) For Playfair’s \(\rightarrow\) cuts
   Assume a straight line intersects one of two parallel lines. If, for contradiction, it doesn’t meet the second parallel, then we would have 2 parallels through the intersection, contradicting Playfair’s, so cuts holds.

b) For cuts \(\rightarrow\) Playfair’s
   Create the parallel \(p\) through \(P\) that is the perpendicular to the perpendicular of \(l\) (using I-12 and I-11 and I-16). Any other line through \(A\) cuts \(p\) so by cuts, it also has to cut \(l\). Thus Playfair’s holds.
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- Proclus assumed that the parallel lines \( l \) and \( m \) were a constant distance apart. What happens in hyperbolic geometry?
  
  In hyperbolic geometry, as distance between \( m \) and any line intersecting it grows, there are examples where it never overtakes the distance between \( m \) and \( l \), which is how we get infinitely many parallels. Sorry Proclus. Good try though!
Lift $P$ and $ABC$. If $ABC$ and $A'B'C'$ are in planes that are not parallel, then the planes intersect in a
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Lift $P$ and $ABC$. If $ABC$ and $A'B'C'$ are in planes that are not parallel, then the planes intersect in a line. Now a side of $ABC$ is on the one plane and the corresponding side of $A'B'C'$ the other, so the intersections of the corresponding sides of the triangles are in both planes and thus on this line. Project.
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- Girard Desargues (1591–1661) explored conics as perspective deformations of a circle and the intersection of parallel lines at infinity.
- Jean-Victor Poncelet (1788–1867) added points and explored properties invariant under projection.
Desargues in Hyperbolic Geometry
Desargues in Spherical Geometry
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Pairs of "parallels" allow the fluid motion to continue undisturbed.

AAA fails for degenerate triangles and gives congruence otherwise.
Pairs of “parallels” allow the fluid motion to continue undisturbed [https://youtu.be/e2kHrDRXzP4](https://youtu.be/e2kHrDRXzP4).

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What is Projective Geometry? A Sphere Divided

- How could you explain division to a young child?
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- Plane: $(x, y) \rightarrow (x + 1, y)$
Surfaces that Locally Look Like the Plane?

- Felix Klein (1849–1925) posed the question in 1890.
- In Klein’s Erlangen Program, the properties of a space were understood by the transformations that preserved them.
- Heinz Hopf’s (1894–1971) rigorous solution was 1925. A complete connected surface which locally looks like the plane is obtained via a quotient by a group of isometries acting without fixed points.
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  Two points are the same if and only if we can get from one point to the other by a transformation: plane, cylinder, Möbius Band, flat Clifford torus, flat Klein bottle.
Surfaces that Locally Look Like the Sphere: $\mathbb{RP}^2$

- Projective Geometry

$\frac{S^2}{\Gamma}$ where $\Gamma = \{ \text{identity, } (x, y, z) \rightarrow (-x, -y, -z) \}$.
Projective Geometry: $\mathbb{RP}^2$

- Elegant
- Duality between points and lines
- Conics
- SAS is fixed (although I-16 still fails)
Hierarchies of Geometries via Transformations

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spherical and hyperbolic $\subset$ projective

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Which do you find most compelling about why Euclid (∼ 325–∼ 265 BCE) wrote the 5th postulate the way he did?

a) Euclid’s 5th might be more of a self-evident truth in nature than Playfair’s as it holds in spherical geometry

b) In Prop 31, Euclid constructs a parallel, but he doesn’t use the language of uniqueness (there can be only one) in Book 1

c) Euclid was trying to keep the same kind of language as the other postulates

d) It is easier to use Euclid’s 5th in the propositions to help prove and support them than it would be in using Playfair’s

e) There was no notion of infinity then, so instead of Playfair’s which refers to never intersecting, Euclid’s 5th gives something constructive about intersection

f) To motivate others to work on parallels and resolve issues he hadn’t and better understand the nature of reality

https://www.youtube.com/watch?v=LPET_HhN0VM
http://cs.appstate.edu/~sjg/class/3610/evals.html