Curvatures and Christoffel Symbols of Modified Brenton Universe and Final Project Recording Ideas

Dr. Sarah's Differential Geometry

Welcoming Environment: Actively listen to others and encourage everyone to participate! Keep an open mind as you engage in our class activities, explore consensus and employ collective thinking across barriers. Maintain a professional tone, show respect and courtesy, and make your contributions matter.

Try to help each other! Discuss and keep track of any questions your group has. Feel free to ask me questions during group work time as well as when I bring us back together.

1. In the article "How to Create Your Own Universe in Three Easy Steps" that we read, Lawrence Brenton created his own universe. I modified Brenton's universe by simply taking a linear increase in the size of the universe via $S(t) = \frac{t}{\pi}$. This still satisfies Brenton's condition that the size of the universe is 0 at time 0 and C at $t = \pi C$, but this does not contract afterwards like Brenton explored. In Maple, using the tensor package, I entered
   coord:=[t, x, y, z];
   and the metric as
   ```
   g_components:=array(symmetric,1..4,1..4,
   [[1,0,0,0],
   [0, -(1+y^2/4)*t^2/Pi^2,1/4*x*y*t^2/Pi^2,-1/2*y*t^2/Pi^2],
   [0,1/4*x*y*t^2/Pi^2,-(1+x^2/4)*t^2/Pi^2,1/2*x*t^2/Pi^2],
   [0,-1/2*y*t^2/Pi^2,1/2*x*t^2/Pi^2,-t^2/Pi^2]]);
   ```
   Next I have Maple compute the matrix inverse g^{ij}, input it as an array for the tensor package, and compute Christoffel symbols and curvatures. The output of the scalar curvature is $-\frac{\pi^2 - 12}{2t^2}$. What is the sign of the scalar curvature?

2. What happens to the scalar curvature as t gets large?

3. Compare with your group and then discuss why does this make some intuitive sense when considering that the size of this universe is growing as t increases?

4. Discuss intuition for what might happen to the scalar curvature of a contracting universe?

5. Recall that the Christoffel symbols are defined in terms of the metric as $\Gamma^a_{bc} = \frac{1}{2} g^{ad}(\partial_b g_{dc} + \partial_c g_{db} - \partial_d g_{bc})$ and that these are used to compute and write equations for curvatures and geodesics. For Γ^2_{21}, what is a, b, and c in Γ^a_{bc}?

6. Substitute a, b and c in but leave d general in $\Gamma^a_{bc} = \frac{1}{2} g^{ad}(\partial_b g_{dc} + \partial_c g_{db} - \partial_d g_{bc})$. Write this out and compare with your group.

7. What do we do with the d? Discuss and respond on `pollev.com/drsarah314`
   a) leave it alone
   b) substitute in $d = 5$
   c) substitute in $d = 1, 2, 3, 4$ one at a time and add
   d) other

8. The metric form and its inverse are

$$g_{ij} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{(1+\frac{y^2}{4})t^2}{\pi^2} & \frac{xyt^2}{4\pi^2} & -\frac{yt^2}{2\pi^2} \\ 0 & \frac{xyt^2}{4\pi^2} & -\frac{(1+\frac{x^2}{4})t^2}{\pi^2} & \frac{xt^2}{2\pi^2} \\ 0 & -\frac{yt^2}{2\pi^2} & \frac{xt^2}{2\pi^2} & -\frac{t^2}{\pi^2} \end{bmatrix} \qquad g^{ij} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{\pi^2}{t^2} & 0 & \frac{y\pi^2}{2t^2} \\ 0 & 0 & -\frac{\pi^2}{t^2} & -\frac{x\pi^2}{2t^2} \\ 0 & \frac{y\pi^2}{2t^2} & -\frac{x\pi^2}{2t^2} & -\frac{\pi^2(x^2+y^2+4)}{4t^2} \end{bmatrix}$$

Is g_{ij} a symmetric matrix where $A = A^T$?

9. Circle $g_{21}, g_{22}, g_{41}, g_{42}$ in the metric form matrix.

10. Circle $g^{21}, g^{22}, g^{23}, g^{24}$ in the inverse matrix.

11. Compute Γ^2_{21} by hand using the direct definition of $\Gamma^a_{bc} = \frac{1}{2}g^{ad}(\partial_b g_{dc} + \partial_c g_{db} - \partial_d g_{bc})$ using #6–#10—be sure to make use of any relevant 0s to avoid writing out other terms you don't need to! Recall that $\partial_1 = \frac{\partial}{\partial \text{first coordinate}} = \frac{\partial}{\partial t}$ since the coordinates are $[t, x, y, z]$ here. Show work.

12. Here is Maple's output of the all the Christoffel symbols, not only Γ^2_{21}.

```
[(1, 1, 1) = 0, (1, 1, 2) = 0, (1, 1, 3) = 0, (1, 1, 4) = 0, (1, 2, 1) = 0,
(1, 2, 2) = ((y^2 + 4)*t)/(4*Pi^2), (1, 2, 3) = -x*y*t/(4*Pi^2),
(1, 2, 4) = y*t/(2*Pi^2), (1, 3, 1) = 0, (1, 3, 2) = -x*y*t/(4*Pi^2),
(1, 3, 3) = ((x^2 + 4)*t)/(4*Pi^2), (1, 3, 4) = -x*t/(2*Pi^2),
(1, 4, 1) = 0, (1, 4, 2) = y*t/(2*Pi^2), (1, 4, 3) = -x*t/(2*Pi^2),
(1, 4, 4) = t/Pi^2, (2, 1, 1) = 0, (2, 1, 2) = 1/t, (2, 1, 3) = 0,
(2, 1, 4) = 0, (2, 2, 1) = 1/t, (2, 2, 2) = 0, (2, 2, 3) = y/4,
(2, 2, 4) = 0, (2, 3, 1) = 0, (2, 3, 2) = y/4, (2, 3, 3) = -x/2,
(2, 3, 4) = 1/2, (2, 4, 1) = 0, (2, 4, 2) = 0, (2, 4, 3) = 1/2,
(2, 4, 4) = 0, (3, 1, 1) = 0, (3, 1, 2) = 0, (3, 1, 3) = 1/t,
(3, 1, 4) = 0, (3, 2, 1) = 0, (3, 2, 2) = -y/2, (3, 2, 3) = x/4,
(3, 2, 4) = -1/2, (3, 3, 1) = 1/t, (3, 3, 2) = x/4, (3, 3, 3) = 0,
(3, 3, 4) = 0, (3, 4, 1) = 0, (3, 4, 2) = -1/2, (3, 4, 3) = 0,
(3, 4, 4) = 0, (4, 1, 1) = 0, (4, 1, 2) = 0, (4, 1, 3) = 0,
(4, 1, 4) = 1/t, (4, 2, 1) = 0, (4, 2, 2) = -x*y/4,
(4, 2, 3) = -y^2/8 + x^2/8, (4, 2, 4) = -x/4, (4, 3, 1) = 0,
(4, 3, 2) = -y^2/8 + x^2/8, (4, 3, 3) = x*y/4, (4, 3, 4) = -y/4,
(4, 4, 1) = 1/t, (4, 4, 2) = -x/4, (4, 4, 3) = -y/4, (4, 4, 4) = 0]
```

Maple outputs Γ^a_{bc} using the notation (a, b, c). Circle Γ^2_{21} in Maple's output.

13. As a review, note that geodesic equations as well as numerous curvatures are defined from the Christoffel symbols and their partial derivatives, building on one another:
Riemann curvature tensor or Riemann-Christoffel tensor $R^a_{bcd} = \partial_c \Gamma^a_{bd} - \partial_d \Gamma^a_{bc} + \Gamma^e_{bd}\Gamma^a_{ec} - \Gamma^e_{bc}\Gamma^a_{ed}$
Ricci tensor $R_{ab} = R^c_{acb}$
Scalar curvature $R = g^{ab}R_{ab}$
Einstein tensor $G_{ab} = R_{ab} - \frac{1}{2}g_{ab}R$

Maple can compute these quickly! Compare your response to #11 with Maple's response and revise if they are different.

14. In your group, discuss whether or not you have recorded a video consisting of slides with you speaking, what you used to do so, and how that went.