Curvature, Torsion, and the Fundamental Theorem of Space Curves

Dr. Sarah's Differential Geometry

Welcoming Environment: Actively listen to others and encourage everyone to participate! Keep an open mind as you engage in our class activities, explore consensus and employ collective thinking across barriers. Maintain a professional tone, show respect and courtesy, and make your contributions matter.

Try to help each other! Discuss and keep track of any questions your group has. Feel free to ask me questions during group work time as well as when I bring us back together.

1. Write the derivatives for the Frenet-Serret equations we have been defining, deriving, and applying:

   $T'(s) =$

   $N'(s) =$

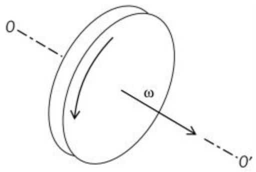
   $B'(s) =$

2. What kind of curve has $\kappa = 0$? Discuss with your group.

3. What kind of curve has $\tau = 0$? Discuss with your group.

4. Give an example of a curve with constant nonzero κ. Discuss with your group.

5. If a coaster car is traveling for a bit on a coaster shaped like:

   http://img.tfd.com/mgh/cep/thumb/Angular-velocity-shown-as-an-axial-vector.jpg

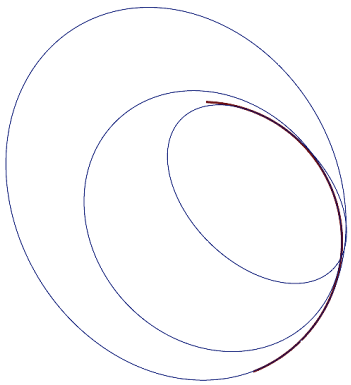
   with the Darboux vector ω, the angular velocity, shown as an axial B vector when $\tau = 0$,

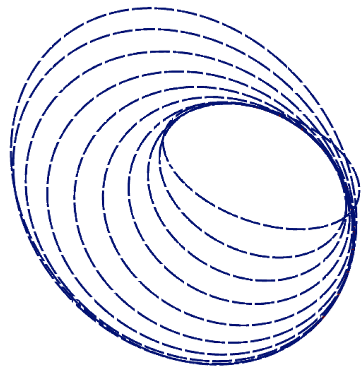
   then how would the people in the car feel the curvature? Discuss with your group.

6. There are numerous different derivations of the fundamental theorem of space curves, that two unit speed curves $\alpha(s)$ and $\beta(s)$ with the same curvature and torsion are themselves the same up to rigid motion in space. The one in our book is left as an exercise, which we'll explore, and makes use of the Frenet-Serret TNB equations as well as rigid motions of 3-space that preserve distance and angle, including translation, rotation and reflection (linear transformations of $\mathbb{R}^3$ in linear algebra). The idea is to first line up the two curves so they begin at the same place in space and their initial Frenet-Serret frames coincide too.

   - Consider $D(s) = T_\alpha(s) \cdot T_\beta(s) + N_\alpha(s) \cdot N_\beta(s) + B_\alpha(s) \cdot B_\beta(s)$. What is $D(0)$, where the curves and their initial frames agree? Discuss with your group.
   - If $D(s) \neq D(0)$, then that gives a measure of the difference of the TNB frames, so let's examine $D'(s)$
     (a) First take the derivative of $D(s)$ by applying the derivative of the dot products.
     (b) Next, substitute the Frenet-Serret derivative equations for $T'_\alpha(s), T'_\beta(s), N'_\alpha(s), N'_\beta(s), B'_\alpha(s), B'_\beta(s)$.
     (c) Collect like terms and apply that $\kappa_\alpha(s) = \kappa_\beta(s)$ and $\tau_\alpha(s) = \tau_\beta(s)$.
     (d) So, what is $D'(s)$? Compare your work with your group.
   - Thus the TNB frames are identical, namely $T(s)$ is, and $\alpha'(s) = T(s) = \beta'(s)$ for all s, so we can integrate to have $\alpha(s)$ and $\beta(s)$ identical up to the integration constant, but we lined up the initial conditions at the starting point, so they are identical up to rigid motion in 3-space!

7. Here is part of the curve of Archytas, possibly the first nonplanar curve studied, with 3 and 10 osculating circles. From these views of the curve of Archytas, what appears to be happening to the curvature as we travel from the bottom of the curve to the top of the curve?





8. Here is a different view of the same curve, oriented differently, and three osculating circles are shown—two almost overlapping. What happens to the torsion as we move from the left side of this view to the right side?