Shape Operator: Mystery Surface, Round Donut and Catalan Surface

Dr. Sarah's Differential Geometry

Welcoming Environment: Actively listen to others and encourage everyone to participate! Keep an open mind as you engage in our class activities, explore consensus and employ collective thinking across barriers. Maintain a professional tone, show respect and courtesy, and make your contributions matter.

Try to help each other! Discuss and keep track of any questions your group has. Feel free to ask me questions during group work time as well as when I bring us back together.

1. For $x(u,v) = (\sqrt{2}u\cos\frac{v}{\sqrt{2}}, \sqrt{2}u\sin\frac{v}{\sqrt{2}}, 0)$, compute $\vec{x}_u$ and $\vec{x}_v$ by-hand and compare with your group.
2. By-hand, compute $E = \vec{x}_u \cdot \vec{x}_u, F = \vec{x}_u \cdot \vec{x}_v, G = \vec{x}_v \cdot \vec{x}_v$ and compare with your group.
3. The metric g_{ij} tells us about what is happening in the tangent plane. Write the metric form ds^2 and the matrix representation.
4. By-hand, compute $\vec{x}_u \times \vec{x}_v$ and compare with your group.
5. What is $U = \dfrac{\vec{x}_u \times \vec{x}_v}{|\vec{x}_u \times \vec{x}_v|}$? Discuss with your group.
6. The shape operator tells us about what is happening normal to the surface by tracking the rates of change of U. We can see how U behaves on any curve on the surface by looking at the shape operator acting on its tangent vectors. Since $\vec{x}_u$ and $\vec{x}_v$ typically form a basis that spans the tangent plane, we compute how U behaves along the coordinate curves by taking the u and v partial derivatives of U (adding a negative sign by convention) and writing them in the basis—so that we can theoretically apply that to any other vector. So, compute $S(\vec{x}_u) = -\nabla_{\vec{x}_u}U = -U_u$.
7. Next compute $S(\vec{x}_v) = -\nabla_{\vec{x}_v}U = -U_v$.
8. Do you recognize this surface from one of the interactive videos? Discuss and then respond on `pollev.com/drsarah314`
   a) yes
   b) unsure or no
9. Consider the top circle on the round physical donut and U on this circle. Does U stay parallel, in which case U_u and U_v are both $\vec{0}$?
10. Consider a vertical circle on the round physical donut and U on this circle. Does U stay parallel, in which case U_u and U_v are both $\vec{0}$ or does U change direction, which means at least one of the partial derivatives is nonzero.
11. The Catalan surface can be parametrized via $x(u,v) = (u - \sin u\cosh v, 1 - \cos u\cosh v, 4\sin\frac{u}{2}\sinh\frac{v}{2})$ and Maple outputs $\vec{x}_u = [1 - \cos u\cosh v, \sin u\cosh v, 2\cos\frac{u}{2}\sinh\frac{v}{2}]$, $\vec{x}_v = [-\sin u\sinh v, -\cos u\sinh v, 2\sin\frac{u}{2}\cosh\frac{v}{2}]$ and $F = \vec{x}_u \cdot \vec{x}_v = 0$. What does $F = 0$ tell us?
12. To see how the shape operator acting on $\vec{x}_v$ is written in terms of $\vec{x}_u$ and $\vec{x}_v$ when they are a basis for the tangent plane, we can set up the augmented matrix with the first column as $\vec{x}_u$, the second column as $\vec{x}_v$ and the equal column as $S(\vec{x}_v) = -\nabla_{\vec{x}_u}U = -U_v$. Using ReducedRowEchelonForm, Maple outputs $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Look again at F. Is this possible or is Maple struggling for the Catalan surface? Discuss.

13. Instead of ReducedRowEchelonForm, if we use GaussianElimination for the same augmented matrix with the first column as $\vec{x}_u$, the second column as $\vec{x}_v$ and the equal column as $S(\vec{x}_v) = -\nabla_{\vec{x}_u} U = -U_v$ and reduce it with respect to trig, the output begins with:

$$\begin{bmatrix}\vec{x}_u & \vec{x}_v & S(\vec{x}_v)\end{bmatrix} \to \begin{bmatrix} 1-\cos u\cosh v & -\sin u \sinh v & \text{too long to put on this handout} \\ 0 & \frac{\sinh v(-\cosh v + \cos u)}{\cos u \cosh v - 1} & \text{too long to put on this handout} \\ 0 & 0 & 0 \end{bmatrix}$$

Discuss what is going on here and then respond on `pollev.com/drsarah314`

a) $S(\vec{x}_v)$ is a linear combination of $\vec{x}_u$ and x_v with nonzero coefficients

b) $S(\vec{x}_v)$ is a linear combination of $\vec{x}_u$ and x_v with one of the coefficients as 0, i.e. a multiple of $\vec{x}_u$ or $\vec{x}_v$

c) $S(\vec{x}_v)$ is not a linear combination of $\vec{x}_u$ and $\vec{x}_v$

d) other

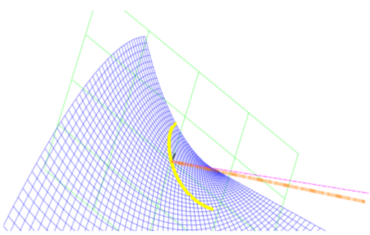
14. Here are two different views related to a curve parametrized by $u = \sin t$ and $v = \cos t$ on the Catalan surface, which I put into Maple for f1 and f2. The image on the left shows part of the surface, the curve, the tangent plane to the surface at a point on the curve, and

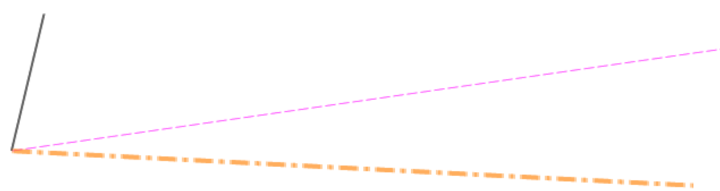
$\vec{\kappa}_\alpha$ as pink dashed thickness 1, the curvature of the curve at the point,

$\vec{\kappa}_n$ as black solid thickness 2, the normal curvature, and

$\vec{\kappa}_g$ tan dashdot style thickness 4, the geodesic curvature.

Label the vectors on both pictures, discuss the visualizations and discuss what these show about the curve.





15. For a different curve with $u = 0$ and $v = t$, what is the parametrization of the curve $\alpha(t)$ on the Catalan surface? Recall that $x(u, v) = (u - \sin u \cosh v, 1 - \cos u \cosh v, 4 \sin \frac{u}{2} \sinh \frac{v}{2})$.

16. By-hand, compute $\alpha'(t)$, $|\alpha'(t)|$, $T(t)$ and $\vec{\kappa}_\alpha$ for the curve you just parameterized.

17. For the same curve, when I execute the evalm(ExtCurv-NormCurv) command, Maple shows that the coordinates of the geodesic curvature vector at a point are $[0., 8.509181280 \times 10^{-10}, 0.]$. Is this curve a geodesic on the Catalan surface? Discuss and then respond on `pollev.com/drsarah314`

a) yes

b) it can be reparametrized to be a geodesic curve with the same torsion and curvature but isn't one currently

c) it is close to a geodesic but isn't one

d) no

18. Search for the Catalan surface on the web and discuss what you find.

19. Search for the mathematician that the Catalan surface is named for on the web and discuss what you find.

20. Roughly sketch a picture of the Catalan surface.