Surface Area and Curvature Matching Activity

Dr. Sarah's Differential Geometry

Welcoming Environment: Actively listen to others and encourage everyone to participate! Keep an open mind as you engage in our class activities, explore consensus and employ collective thinking across barriers. Maintain a professional tone, show respect and courtesy, and make your contributions matter.

Try to help each other! Discuss and keep track of any questions your group has. Feel free to ask me questions during group work time as well as when I bring us back together.

1. Given the radius r cylinder $x(\theta, z) = (r\cos\theta, r\sin\theta, z)$ where r is constant, compute E, F, and G by hand.

2. Set up the double integral for surface area of a cylinder with height from $z = 0$ to h and limits of integration for θ too—discuss what they should be.

3. Compute this double integral by hand.

4. Sketch a picture of this finite cylinder and label the limits for the z and θ values on it as well as r.

5. Sketch a picture of the first sheet of the covering of this cylinder and label the base and height of this covering in terms of z, θ, and r.

6. Does the surface area of the covering match the surface area from the integration of the square root of the determinant of the metric form? Discuss.

7. For the Catalan surface, Maple outputs

   $E = (4\cosh^2\frac{v}{2} - 4)\cos^2\frac{u}{2} - 2\cos u\cosh v + \cosh^2 v + 1$

   $F = 0$

   $G = (-4\cos^2\frac{u}{2} + 4)\cosh^2\frac{v}{2} + \cosh^2 v - 1$

   To show $E = G$ (this isn't usually the case but it is here!) we'll expand E, apply the double angle formulas for both $\cos x$ and $\cosh x$, expand and reduce:

   $E = (4\cosh^2\frac{v}{2} - 4)\cos^2\frac{u}{2} - 2\cos u\cosh v + \cosh^2 v + 1$
   $= 4\cosh^2\frac{v}{2}\cos^2\frac{u}{2} - 4\cos^2\frac{u}{2} - 2\cos u\cosh v + \cosh^2 v + 1$
   $= 4\cosh^2\frac{v}{2}\cos^2\frac{u}{2} - 4\cos^2\frac{u}{2} - 2(2\cos^2\frac{u}{2} - 1)(2\cosh^2\frac{v}{2} - 1) + \cosh^2 v + 1$
   $= 4\cosh^2\frac{v}{2}\cos^2\frac{u}{2} - 4\cos^2\frac{u}{2} - 8\cosh^2\frac{v}{2}\cos^2\frac{u}{2} + 4\cos^2\frac{u}{2} + 4\cosh^2\frac{v}{2} - 2 + \cosh^2 v + 1$

   Examine the steps, considering how the double angle formula and foiling came in to play, and then reduce by combining like terms

   $=$

   Next, compare to G.

8. Since $E = G$ and $F = 0$ then $\sqrt{EG - F^2} = G$ when G is nonnegative, which it is here since $0 \leq \cos^2 x \leq 1$ and $\cosh^2 x \geq 1$. Cap it off with $u = 0$ to 2π and $v = 0$ to 2π as you set up but do not solve the double integral for finite surface area and include the limits of integration.

9. Follow the instruction below to match items and then label matches as intrinsic, scalars, and names:

   - Match a curvature symbol to a formula and to the physical/geometric description (see below).
   - Next, which are extrinsic curvatures and which are intrinsic curvatures? Label these.
   - Next, which are vectors and which are scalars? Label these.
   - If the curvature is not already named, what is its name? Label any that aren't already named.
   - Which does this expression connect with?

$$\frac{1}{(EG-F^2)^2}\left(\begin{vmatrix} -\frac{E_{vv}}{2}+F_{uv}-\frac{G_{uu}}{2} & \frac{E_u}{2} & F_u-\frac{E_v}{2} \\ F_v-\frac{G_u}{2} & E & F \\ \frac{G_v}{2} & F & G \end{vmatrix} - \begin{vmatrix} 0 & \frac{E_v}{2} & \frac{G_u}{2} \\ \frac{E_v}{2} & E & F \\ \frac{G_u}{2} & F & G \end{vmatrix}\right)$$

   Curvature symbols (I've left off vector notation on purpose to make the identification of scalars more interesting):

   - κ_1
   - κ_2
   - K
   - H
   - κ_α
   - κ_n
   - κ_g

   Formulas:

   - $S_p(\vec{w}) = \kappa_1 \vec{w}$
   - $\frac{\kappa_1+\kappa_2}{2} = \frac{lG-2mF+nE}{2(EG-F^2)}$
   - $\kappa_\alpha - \kappa_n$
   - $S_p(\vec{w}) = \kappa_2 \vec{w}$
   - $(U \cdot \kappa_\alpha)U$
   - $\kappa_1\kappa_2 = \frac{ln-m^2}{EG-F^2} = \frac{|II|}{|I|}$
   - $\dfrac{T'(t)}{|\alpha'(t)|}$

   Physical/Geometric Descriptions

   - curvature vector of a curve
   - normal curvature components of a curve
   - tangential curvature components of a curve
   - maximum normal curvature at a point
   - minimum normal curvature at a point
   - some measure of how a surface bends at a point with respect to T_pM
   - some measure of whether a surface can be area minimizing and a soap bubble

10. Review how we showed that the square root of the determinant of the metric form gives the area of a flat $\vec{x}_u, \vec{x}_v$ parallelogram in the interactive video.