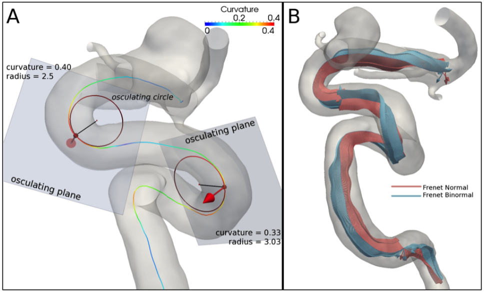
Applications of Curvature and Torsion

In order to make my clock even more exact.... I never had expected I would discover, I have now hit upon, the undoubtedly true shape of curves... I determined it by geometric reasoning. (Christiaan Huygens Dec. 1659)

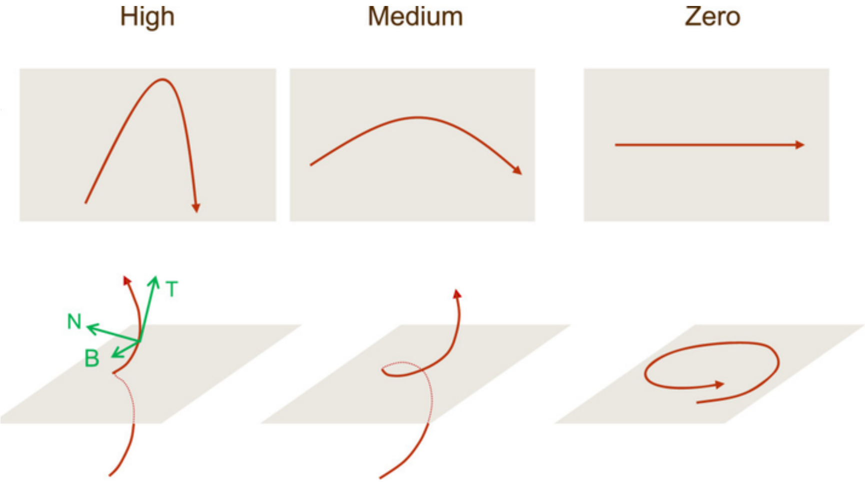


Piccinelli, Marina et al. 2009. "A framework for geometric analysis of vascular structures" *IEEE Trans. Med. Imaging*

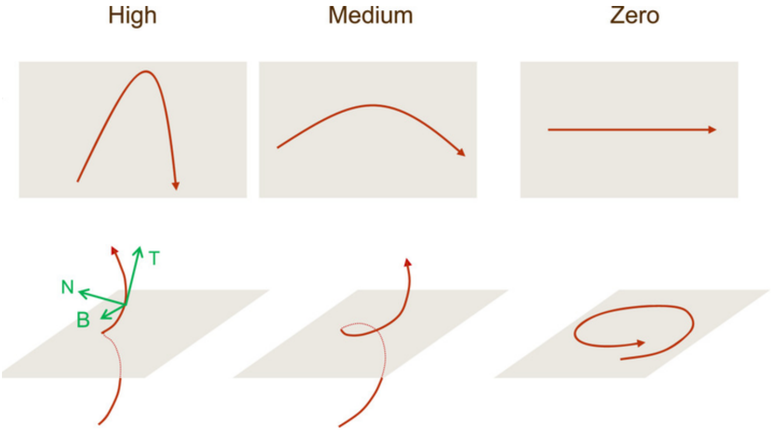
$T, \vec{\kappa}, \kappa, N, B, \tau, T', N', B'$

- $T(t) = \frac{\alpha'(t)}{|\alpha'(t)|}$
- $\vec{\kappa} = \frac{T'(t)}{|\alpha'(t)|}$
- $\kappa = |\vec{\kappa}|$
- $N(t) = \frac{\vec{\kappa}}{|\vec{\kappa}|}$
- $B(t) = T \times N$
- τ: compute $\frac{B'(t)}{|\alpha'(t)|}$ & compare it to N (they are multiples of each other) to find $-\tau$ and then τ
- $T'(s) = \kappa N$
  $N'(s) = -\kappa T + \tau B$
  $B'(s) = -\tau N$
  $$\begin{bmatrix} T'(s) \\ N'(s) \\ B'(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}$$

- Prove that $\alpha(s)$ with $\kappa = 0$ is a line
- Prove that $\alpha(s)$ with $\tau = 0$ is a planar curve
- Prove that $\alpha(s)$ planar with $\kappa > 0$ constant is circular



Pitt et al.: "Polyphony: superposition independent methods for ensemble-based drug discovery." *BMC Bioinformatics* 2014 15:324.



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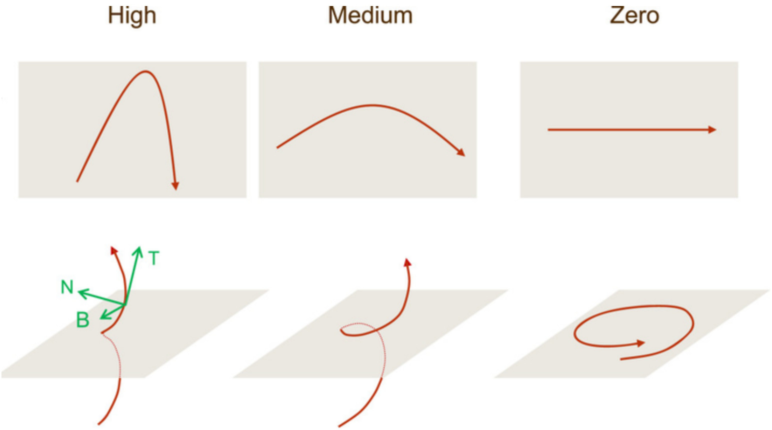
To prove that $\alpha(s)$ with $\kappa = 0$ is a line, assume $\kappa = 0$. Then $T'(s) =$



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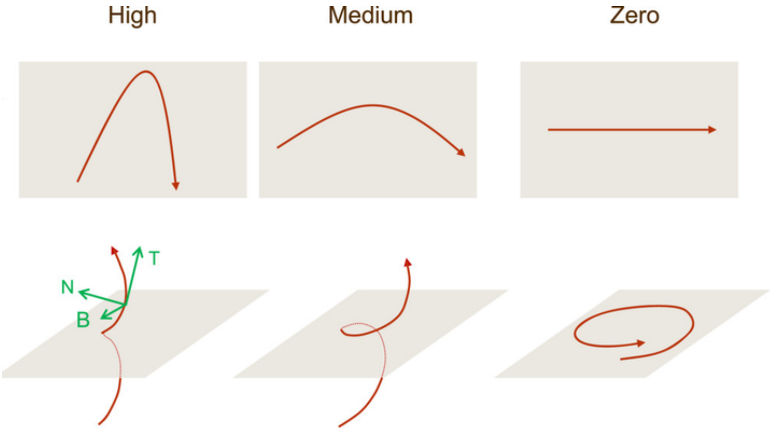
To prove that $\alpha(s)$ with $\kappa = 0$ is a line, assume $\kappa = 0$. Then $T'(s) = \kappa N = 0N = \vec{0}$.



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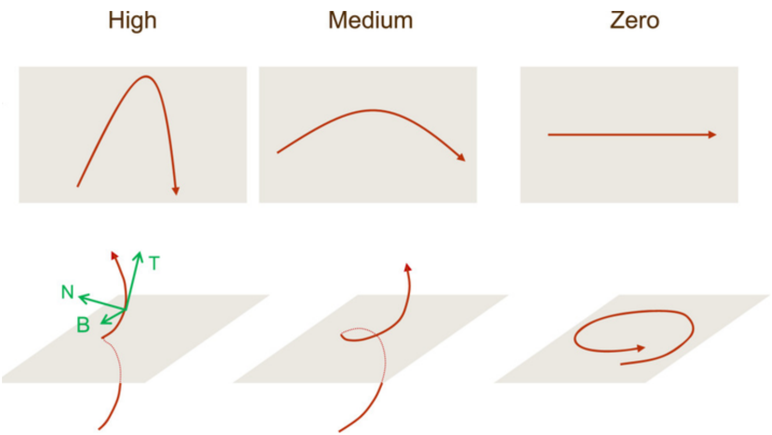
To prove that $\alpha(s)$ with $\kappa = 0$ is a line, assume $\kappa = 0$. Then $T'(s) = \kappa N = 0N = \vec{0}$. So $T(s) = \int T'(s)ds$ is a constant, call it $\vec{v}$.



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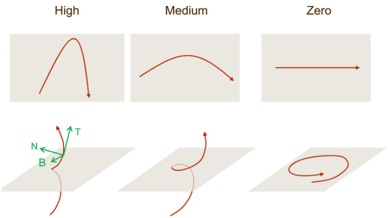

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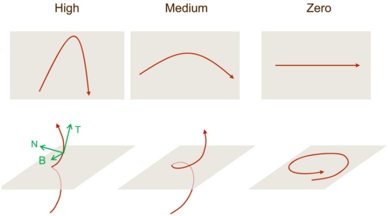

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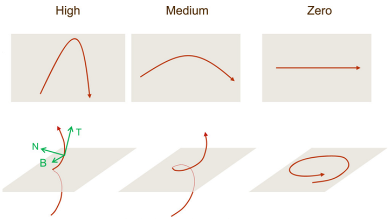
To prove that $\alpha(s)$ with $\tau = 0$ is a planar curve, assume $\tau = 0$. Now $B' = -\tau N = 0N = \vec{0}$. So B is constant. Examine the plane determined by $\alpha(0)$ and B: $((x, y, z) - \alpha(0)) \cdot B = 0$. To show that $\alpha(s)$ is inside of it for all s, consider $(\alpha(s) - \alpha(0)) \cdot B$.



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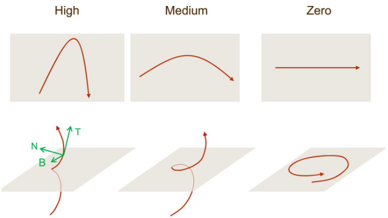
$(\alpha'(s) - \vec{0}) \cdot B + (\alpha(s) - \alpha(0)) \cdot B' = \alpha'(s) \cdot B = T \cdot B$



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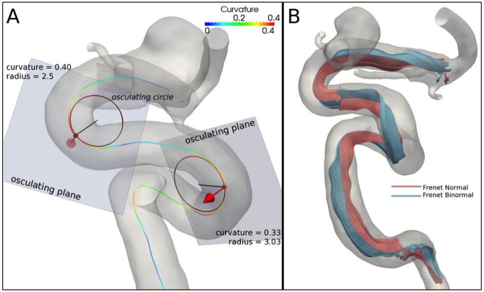

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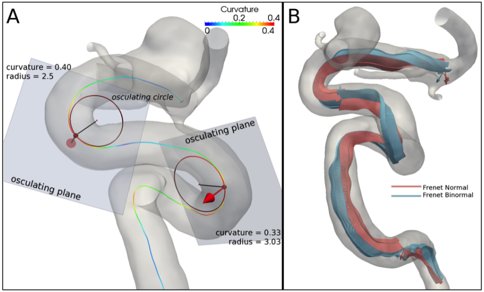
To show $k > 0$ constant for a plane curve then $\alpha(s)$ is part of a circle, assume $k > 0$ constant for a plane curve. Look at $\alpha(s) + \frac{1}{\kappa}N(s)$



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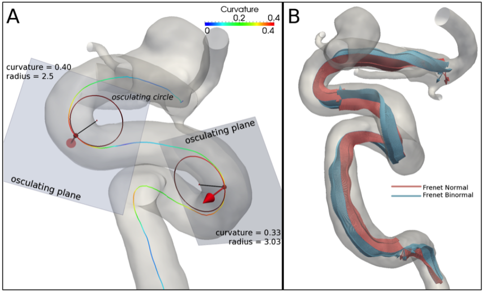
$=$



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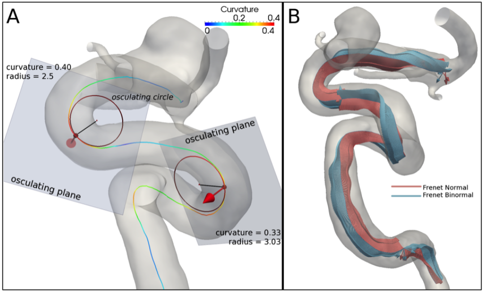
$= T(s) + \frac{1}{\kappa}(-\kappa T + \tau B) = T(s) + \frac{1}{\kappa}(-\kappa T + 0B) = T(s) - T(s) = \vec{0}$.



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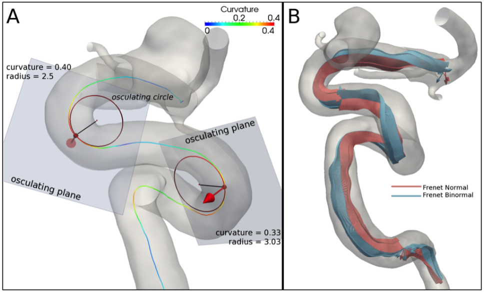

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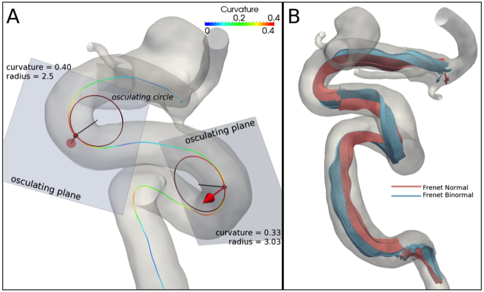

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The equation of a circle in 3-space is the intersection of $|(x, y, z) -$ fixed center$| = r$ with a fixed plane. Now $|\alpha(s) - (\alpha(s) + \frac{1}{\kappa}N(s))| = |\frac{1}{\kappa}N(s)| = \frac{1}{\kappa}$, so $\alpha(s)$ is part of that circle with fixed center $\alpha(s) + \frac{1}{\kappa}N(s))$ and fixed radius $\frac{1}{\kappa}$. □

Darboux Vector $\omega(s)$: Angular Velocity Vector

rigid body translation and rotation along a nonlinear curve

$$T'(s) = \omega(s) \times T(s), \quad N'(s) = \omega(s) \times N(s), \quad B'(s) = \omega(s) \times B(s)$$

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Write $\omega = c_1 T + c_2 N + c_3 B$. To find c_2, notice
$\kappa N = T' = \omega(s) \times T(s)$, so

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Write $\omega = c_1 T + c_2 N + c_3 B$. To find c_2, notice
$\kappa N = T' = \omega(s) \times T(s)$, so $\kappa N \perp \omega$. But then
$0 = \omega \cdot \kappa N = (c_1 T + c_2 N + c_3 B) \cdot \kappa N$
$= c_1 T \cdot \kappa N + c_2 N \cdot \kappa N + c_3 B \cdot \kappa N = c_2 \kappa N \cdot N = c_2 \kappa$.

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So $c_2 = 0$ and $\omega = c_1 T + c_3 B$.

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$-\kappa T + \tau B = N' = \omega(s) \times N(s)$ so $-\kappa T + \tau B \perp \omega$. But then
$0 = \omega \cdot (-\kappa T + \tau B) = (c_1 T + c_3 B) \cdot (-\kappa T + \tau B)$
$= -\kappa c_1 T \cdot T + \tau c_1 T \cdot B - \kappa c_3 B \cdot T + \tau c_3 B \cdot B = -\kappa c_1 + \tau c_3$

$c_1 = \tau$ and $c_3 = \kappa$ and $\omega(s) = \tau T + \kappa B$

Darboux Vector $\omega(s)$: Angular Velocity Vector

$$\omega(s) = \tau T + \kappa B$$

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