Project 4: Intrinsic Geometry of Cones

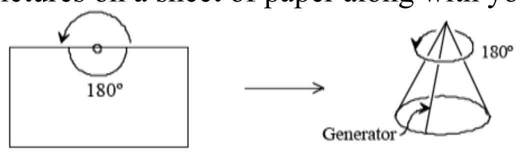
You may work with at most **one** other person and turn in one per group.

You should be prepared to present any of the problems your group turned in, if called on to do so. Each group must write up their work in their own words. 4140 students complete all but the last problem - graduate students in 5530 complete all of them.

The purpose of homework is to learn and practice hands-on strategies, concepts, and develop critical thinking and problem-solving skills, so you should try problems on your own. Feel free to talk to me or each other if you are stuck on this assignment, but be sure to acknowledge any sources including each other, like “The insight for this solution came from a conversation with Joel.” If you know how to do a problem and are asked for help, try to give hints rather than the solution: Give a man a fish and you feed him for a day. Teach a man to fish and you feed him for a lifetime (or at least this course)

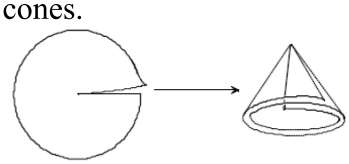
Be sure to define all terms you use, draw your own pictures, explain in your own words, give proper reference. You will have 3 models to turn in along with your work.

Make a model of a 180-degree cone and use it to answer the following questions. Turn in your model and in addition, explain your work and your answers by drawing your own pictures on a sheet of paper along with your text explanations.



180-degree cone

1) What are the geodesics on the surface of a 180-degree cone? Why? Have you listed all of them? How do you know?
2) How many geodesics join two points on a 180-degree cone? Is there always at least one?
3) On a 180-degree cone, can a geodesic ever intersect itself? How many times?
4) Would your answer to questions 2 change if the angle of the cone varied between 0 < cone angle < 360? Make a cone where you can vary the angle and experiment to turn in. If your answer would change, give an example of differing behavior in a specific cone. If not, explain why the behavior would be the same for all such cones.

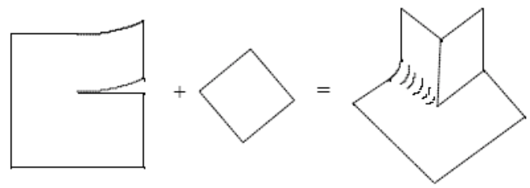


variable angle (0-360 degrees) cone

5) Would your answer to question 3 differ if the angle of the cone varied between 0 < cone angle < 360? If so, give an example of differing behavior in

a specific cone. If not, explain why the behavior would be the same for all such cones.

Notice that 450-degree cones appear commonly in buildings as so-called "outside corners." Make a model of a 450-degree cone and use it to answer the following questions and to turn in your model. In addition, explain your work and your answers by drawing pictures along with your text explanations.



450-degree Cone

6) How many geodesics join two points on a 450-degree cone? Is there always at least one? Explain.
7) On a 450-degree cone, can a geodesic ever intersect itself? How many times?
8) On a 450-degree cone, find a point P (other than the cone point) and a geodesic *l* (not through the cone point) such that there are many geodesics through P that do not intersect *l*. Sketch a picture and compare this situation to the usual parallel postulate for the plane.

Geodesic polar coordinates on an α-degree cone can be described intrinsically by $y(\theta,r)$={the point p on the cone, where r is the length of the line segment from p to the cone point and θ is the angle along the surface between this segment and a fixed reference ray from the cone point}. These coordinates work for any cone, even those with cone angle larger than 360 degrees.

9) Show that if a geodesic is on the cone and $p=(\beta, d)$ is the point on that geodesic closest to the cone point, then an arbitrary point $y(\theta,r)$ on the geodesic satisfies the equation $r=d\sec(\theta-\beta)$. Hint: draw a picture that represents this situation in the cone and in the covering plane, and apply trigonometry.
10) (**Extra credit for undergraduates, required for graduate students**) An equation for a geodesic on a cone in terms of extrinsic local coordinates is $x(\theta,r)=(r\sin\varphi\cos(\frac{2\pi\theta}{\alpha}), r\sin\varphi\sin(\frac{2\pi\theta}{\alpha}), r\cos\varphi)$, where φ is the angle between the axis of the cone and a generator of the cone, $r=\sqrt{d^2+s^2}$, $\theta=\beta$ +arctan (s/d), and s = arclength along the geodesic. Use the equation for θ in order to argue how many times a geodesic on a cone of angle α intersects itself. How does the number of self-intersections depend on the cone angle?