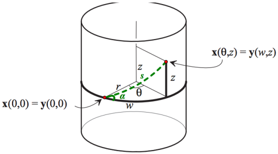
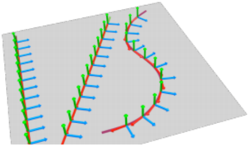
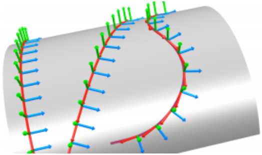
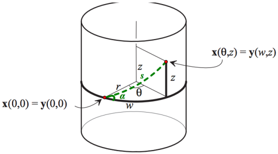
The Speed v of a Geodesic: Lemma 5.1.7 p. 212

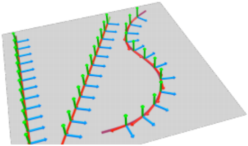


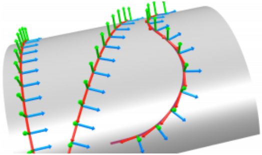



Adapted `http://pi.math.cornell.edu/~henderson/courses/M4540-S12/11-DG-front+Ch1.pdf`

The Speed v of a Geodesic: Lemma 5.1.7 p. 212



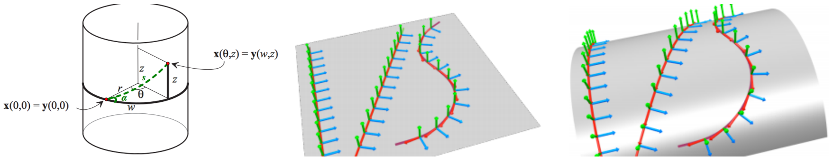




Adapted `http://pi.math.cornell.edu/~henderson/courses/M4540-S12/11-DG-front+Ch1.pdf`

Don't feel any curvature in the tangent plane T_pSurface – only normal to the surface (toy car)

The Speed v of a Geodesic: Lemma 5.1.7 p. 212



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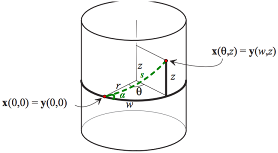
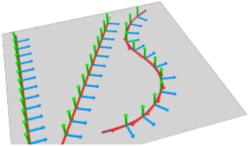
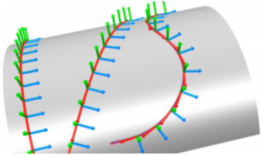
Don't feel any curvature in the tangent plane T_pSurface – only normal to the surface (toy car)

$v = |\alpha'(t)| = |\vec{v}|, \qquad T(t) = \frac{\alpha'(t)}{|\alpha'(t)|} = \frac{\alpha'(t)}{v(t)}$ so $\alpha'(t) = v(t)T(t)$

$\alpha''(t)$

The Speed v of a Geodesic: Lemma 5.1.7 p. 212



Adapted `http://pi.math.cornell.edu/~henderson/courses/M4540-S12/11-DG-front+Ch1.pdf`

Don't feel any curvature in the tangent plane T_pSurface – only normal to the surface (toy car)

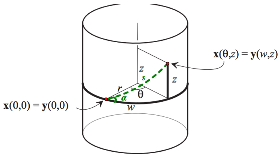
$v = |\alpha'(t)| = |\vec{v}|, \qquad T(t) = \frac{\alpha'(t)}{|\alpha'(t)|} = \frac{\alpha'(t)}{v(t)}$ so $\alpha'(t) = v(t)T(t)$

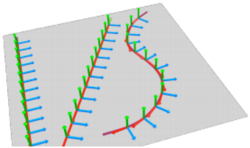
$\alpha''(t) = v'(t)T(t) + v(t)T'(t)$

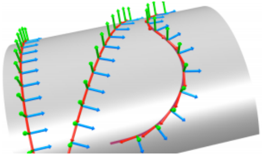
$v'(t)$: linear or tangential acceleration (tangential component of acceleration vector)

For a geodesic, since we don't feel any curvature in the tangent plane—only normal to the surface—

The Speed v of a Geodesic: Lemma 5.1.7 p. 212







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Don't feel any curvature in the tangent plane T_pSurface – only normal to the surface (toy car)

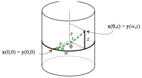
$v = |\alpha'(t)| = |\vec{v}|, \qquad T(t) = \frac{\alpha'(t)}{|\alpha'(t)|} = \frac{\alpha'(t)}{v(t)}$ so $\alpha'(t) = v(t)T(t)$

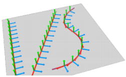
$\alpha''(t) = v'(t)T(t) + v(t)T'(t)$

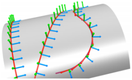
$v'(t)$: linear or tangential acceleration (tangential component of acceleration vector)

For a geodesic, since we don't feel any curvature in the tangent plane—only normal to the surface— $v'(t) = 0$ so v is constant.

Recognizing Geodesics on Cylinder using $\vec{\kappa}_\alpha, \vec{\kappa}_n, \vec{\kappa}_g$



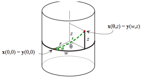


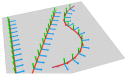
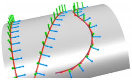


Adapted http://pi.math.cornell.edu/~henderson/courses/M4540-S12/11-DG-front+Ch1.pdf

$x(u,v) = (cos(u), sin(u), v)$ Normal U to the surface?

Recognizing Geodesics on Cylinder using $\vec{\kappa}_\alpha, \vec{\kappa}_n, \vec{\kappa}_g$



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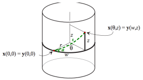
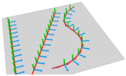
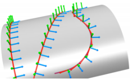
$U = \frac{\vec{x}_u \times \vec{x}_v}{|\vec{x}_u \times \vec{x}_v|} = (cos(u), sin(u), 0)$

Ex 1: $\alpha(t) = (cos(t), sin(t), sin(t))$.

Recognizing Geodesics on Cylinder using $\vec{\kappa}_\alpha, \vec{\kappa}_n, \vec{\kappa}_g$



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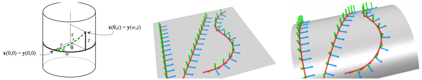
Ex 1: $\alpha(t) = (cos(t), sin(t), sin(t))$. Then $\alpha'(t) = (-sin(t), cos(t), cos(t))$ and the speed is $\sqrt{1 + cos^2(t)}$, which is not constant, so α can't possibly be a geodesic.

Recognizing Geodesics on Cylinder using $\vec{\kappa}_\alpha, \vec{\kappa}_n, \vec{\kappa}_g$



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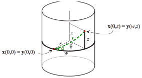
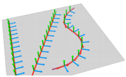
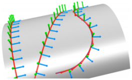
Recognizing Geodesics on Cylinder using $\vec{\kappa}_\alpha, \vec{\kappa}_n, \vec{\kappa}_g$



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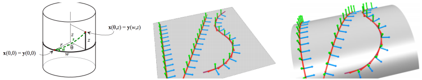
Recognizing Geodesics on Cylinder using $\vec{\kappa}_\alpha, \vec{\kappa}_n, \vec{\kappa}_g$



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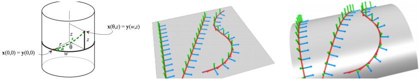
$\vec{\kappa} = \frac{T'(t)}{\sqrt{1+cos^2(t)}}$ will require quotient rule or similar and certainly felt by the bug because it is not only in the U direction

Ex 2: $\gamma(t) = (cos(t), sin(t), t)$ Calculate $\vec{\kappa} = \frac{T'(t)}{|\gamma'(t)|}$ and compare with U to explain why it isn't felt by the bug

Recognizing Geodesics on Cylinder using $\vec{\kappa}_\alpha, \vec{\kappa}_n, \vec{\kappa}_g$



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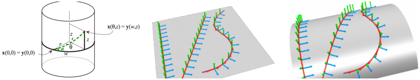
$U = \frac{\vec{x}_u \times \vec{x}_v}{|\vec{x}_u \times \vec{x}_v|} = (cos(u), sin(u), 0)$

$\vec{\kappa}_\alpha$ (curve's curvature vector): $\frac{T'(t)}{|\alpha'(t)|}$

$\vec{\kappa}_n$ (normal curvature): projection of $\vec{\kappa}_\alpha$ onto $U = (U \cdot \vec{\kappa}_\alpha)U$

$\vec{\kappa}_g$ (geodesic curvature): $\vec{\kappa}_\alpha$ - $\vec{\kappa}_n$

Recognizing Geodesics on Cylinder using $\vec{\kappa}_\alpha, \vec{\kappa}_n, \vec{\kappa}_g$



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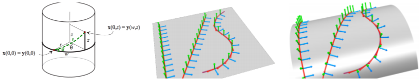
Ex 3: $\gamma(t) = (cos(t), sin(t), 0)$ is a geodesic.

Recognizing Geodesics on Cylinder using $\vec{\kappa}_\alpha, \vec{\kappa}_n, \vec{\kappa}_g$



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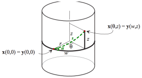
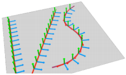
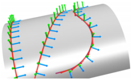
$\frac{\gamma'(t)}{|\gamma'(t)|} = T = (-sin(t), cos(t), 0)$ (speed is 1).

$\vec{\kappa} = \frac{T'(t)}{|\gamma'(t)|} = (-cos(t), -sin(t), 0)$ no T_pM component, only U

Recognizing Geodesics on Cylinder using $\vec{\kappa}_\alpha, \vec{\kappa}_n, \vec{\kappa}_g$



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Ex 3: $\gamma(t) = (cos(t), sin(t), 0)$ is a geodesic.

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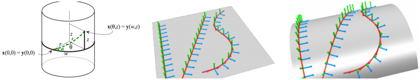
$\vec{\kappa} = \frac{T'(t)}{|\gamma'(t)|} = (-cos(t), -sin(t), 0)$ no T_pM component, only U

Ex 4: $\gamma(t) = (cos(0), sin(0), t)$ is a geodesic.

Recognizing Geodesics on Cylinder using $\vec{\kappa}_\alpha, \vec{\kappa}_n, \vec{\kappa}_g$



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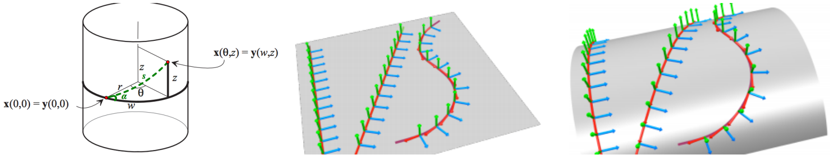
Classifying Cylinder Geodesics Using α''



Adapted `http://pi.math.cornell.edu/~henderson/courses/M4540-S12/11-DG-front+Ch1.pdf`

surface $x(u, v) = (cos(u), sin(u), v)$—two free variables u, v

$\vec{x}_u = (-sin(u), cos(u), 0), \vec{x}_v = (0, 0, 1)$.

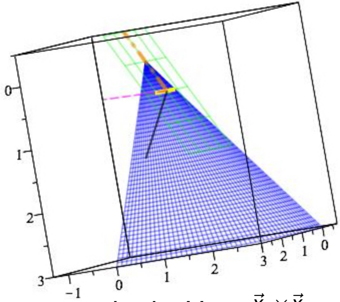
$U = \frac{\vec{x}_u \times \vec{x}_v}{|\vec{x}_u \times \vec{x}_v|} = (cos(u), sin(u), 0)$

curve on surface $\alpha(t) = (cos(u(t)), sin(u(t)), v(t))$— t free

geodesic will have no tangential components of $\alpha''(t)$

Maple File on Geodesic and Normal Curvatures

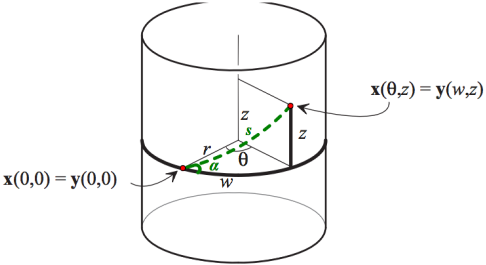
adapted from David Henderson



$\vec{\kappa}_\alpha$ pink dashed thickness 1
$\vec{\kappa}_n$ black solid thickness 2
$\vec{\kappa}_g$ tan dashdot style thickness 4

- The *unit normal* to the surface at a point is $U = \frac{\vec{x}_u \times \vec{x}_v}{|\vec{x}_u \times \vec{x}_v|}$
- If $\vec{\kappa}_\alpha$ is the curvature vector for a curve $\alpha(t)$ on the surface then the *normal curvature* is the projection onto U:
$$\vec{\kappa}_n = (U \cdot \vec{\kappa}_\alpha)U$$
- The *geodesic curvature* is what is felt by the bug (in the tangent plane T_pM): $\vec{\kappa}_g = \vec{\kappa}_\alpha - \vec{\kappa}_n$

Intrinsic Coordinates on a Cylinder



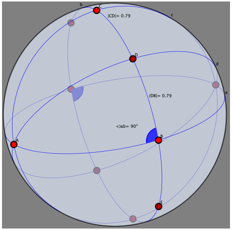
Adapted http://pi.math.cornell.edu/~henderson/courses/M4540-S12/11-DG-front+Ch1.pdf

- Choose (0,0) as an intrinsic origin. There is 1 geodesic that will return there, so call that the base curve
- Choose +z as a direction $\perp$ to the base curve

Geodesic rectangular coordinates: $y(w, z)$ = walk w units along base curve and turn 90° to positive z and travel z units.

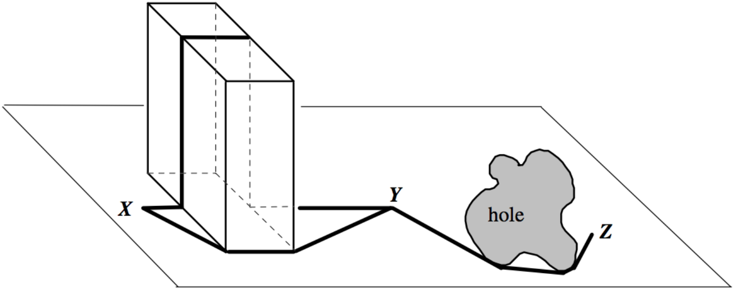
Geodesic polar coordinates: $y(\alpha, s)$ = turn α degrees from the base curve and walk s units along that geodesic

(Intrinsically Straight) Geodesics on a Sphere



- symmetry and our feet, rolling a ball in a line of paint, but can't flatten a sphere with a C^2 isometry

1. Is a latitude a geodesic?
2. How many differently shaped geodesics can you find?
3. Can a geodesic ever intersect itself? Why?
4. Is straight always shortest distance? Explain.
5. Is shortest distance always straight? Explain.
6. How many geodesics join 2 points?



David Henderson: http://pi.math.cornell.edu/~dwh/papers/EB-DG/EB-DG-web.htm

If a surface is smooth (in the C^2 sense, with local coordinates whose first and second derivatives exist and are continuous), then a geodesic on the surface is always the locally shortest path between “nearby” points. If the surface is also geodesically complete (that is, every geodesic on it can be extended indefinitely, for example, there are no holes), then any two points can be joined by a geodesic that is the shortest path.