Homework 4: Intrinsic Geometry of Cones

You may work alone or in a group of up to 2 people and turn in one per group. Each group must write up their work in their own words and give any credit to others where it is due. Try all the problems, but carefully write up to turn in this number:
4140 students complete all of the first 4 problems. Graduate students complete all 5.

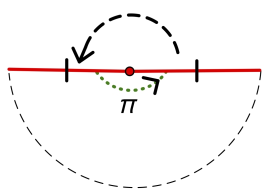
The purpose of homework is to learn and practice computational strategies, concepts, and develop critical thinking and problem-solving skills, so you should try problems on your own. Feel free to talk to me or each other if you are stuck on this assignment, but be sure to acknowledge any sources including each other, except your group members or me. If you know how to do a problem and are asked for help, try to give hints rather than the solution.

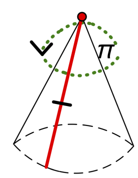
Instructions:

–draw or create your own pictures to go along with your responses and/or annotate your models
–explain in your own words and give any references you used, including each other

1. π cone

   a) A cone can be thought of as a generator from a cone point that sweeps out the cone angle, as with red below. Make a physical model of a π cone and write your name(s) on it to turn in. Notice you can wrap twice from the two unique sheets in the covering:





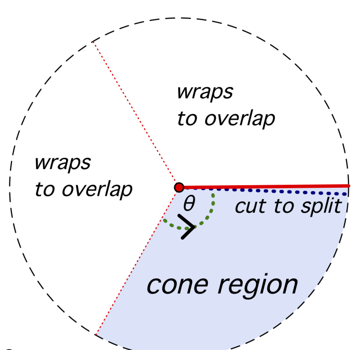
   b) Straight lines in the covering map to geodesics in the cone so how many geodesics join two points on a π cone? Why? Is there always at least one? For each of the questions in this homework, including this one, draw your own pictures (here draw these in the 2-sheeted covering or the cone and/or annotate your model) to go along with your text annotations/explanations.

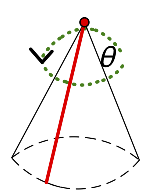
   c) On a π cone, can a geodesic ever intersect itself? Why or why not? If so, how many times?

   d) What are the different kinds of geodesics on the surface of a π cone? Why?

2. $0 <$ cone angle $< 2\pi$ variable cone

   a) Make a cone where you can vary the angle and write your name(s) on it to turn in.





   $\frac{2\pi}{3}$ cone. We can vary the angle by changing the cone region before we wrap the rest around.

   b) Does your response to 1b) change if the angle of the cone varies so that $0 <$ cone angle $< 2\pi$? Either give an example of differing behavior or explain why it would be the same.

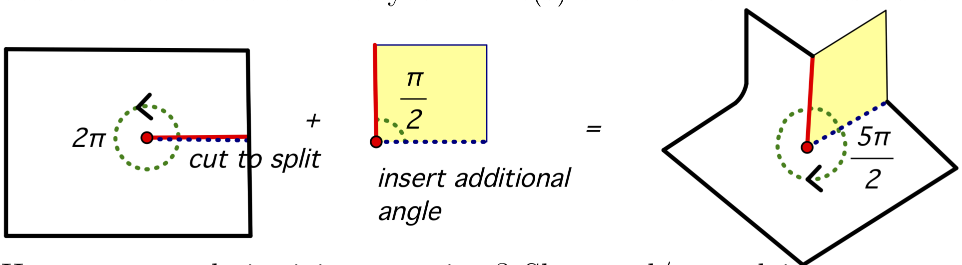
c) Does your response to 1c) change if the angle of the cone varies so that $0 <$ cone angle $< 2\pi$? Either give an example of differing behavior or explain why it would be the same.

d) What happens to 1b) and 1c) as the cone angle gets close to 0? Show and/or explain.

e) What happens to 1b) and 1c) as the cone angle gets close to 2π? Show and/or explain.

3. $\frac{5\pi}{2}$ cone (appears commonly in buildings as so-called "outside corners.")

a) Make a $\frac{5\pi}{2}$ cone by taping in the extra $\frac{\pi}{2}$ piece to a rectangular sheet with red to red and dashed to dashed and write your name(s) on it to turn in. It is its own covering (no sheets).



b) How many geodesics join two points? Show and/or explain.

c) Is there always at least one geodesic between any two points? Show and/or explain.

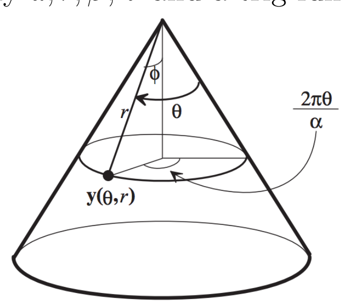
d) Can a geodesic ever intersect itself? Show and/or explain. If so, how many times?

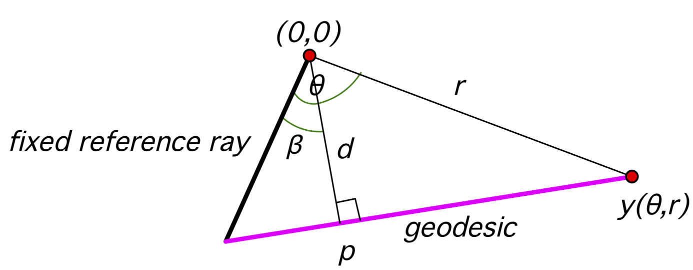
e) Imagine the cone extends indefinitely in all directions. Find a point P (other than the cone point) and a geodesic l (not through the cone point) such that there is more than one geodesic through P that do not intersect l. Sketch a picture and compare this situation to the usual parallel postulate for the plane.

4. Geodesic polar coordinates for the cone can be described intrinsically. Here, r is the length of the line segment from the point $y(\theta, r)$ on a geodesic to the cone point and θ is the angle along the surface between this segment and a fixed reference ray from the cone point, as shown in the cone and covering diagrams. These coordinates work for any cone, even those with cone angle larger than 2π. Let $p = (\beta, d)$ be the point on that geodesic closest to the cone point.

a) At $p = (\beta, d)$, why is the angle the geodesic makes with the line to the cone point $\frac{\pi}{2}$?

b) Apply trigonometry to write down an equation that $y(\theta, r)$ on the geodesic satisfies involving only d, r, β, θ and a trig function.





5. **Graduate Problem**: In extrinsic coordinates in $\mathbb{R}^3$, an equation for a geodesic on a cone of angle α is $x(\theta, r) = (r \sin\phi \cos(\frac{2\pi\theta}{\alpha}), r \sin\phi \sin(\frac{2\pi\theta}{\alpha}), r\cos\phi)$, where some of the variables are already defined above in problem 4 and ϕ is the angle between the axis of the cone and a fixed reference ray. In addition, $\theta = \beta + \arctan(\frac{s}{d})$, where s is the arclength along the geodesic from the fixed reference ray and $r = \sqrt{d^2 + s^2}$. Use this equation for θ to argue how many times a geodesic on a cone of angle α intersects itself. How does the number of self-intersections depend on the cone angle?