Carolyn Gordon
1950-present

Hearing the Shape of a Drum
First Draft 3/02
by Dr. Sarah

Introduction

Carolyn Gordon is a well-known and respected geometer. While she is best known for her groundbreaking work on hearing the shape of a drum, she continues to do research as a leader in the related field of spectral geometry. She presents her research at conferences all over the world and has received numerous grants and awards. In this classroom worksheet, we will discuss issues related to gender in her experiences, her mathematical style, and will explore some of her research on hearing the shape of a drum.

Influences, Support and Gender Issues

Carolyn Gordon's older sister was the biggest influence on her choice to become a mathematician. While her sister is not a mathematician, she loves mathematics because it is like a puzzle. As a child, Carolyn Gordon looked up to her sister and then discovered that she also liked mathematics. In addition, because her older sister enjoyed mathematics, she was a role model for her, showing her that it was acceptable for a girl to like mathematics. Her family expected her to attend college, then get married and have a family instead of a career. These were common expectations during the time that she grew up. On the other hand, her family did support her decision to obtain a PhD.

At times she had the support of society while at other times she didn't seem to have this support. For example, as a girl, she was teased by a group of boys. A competitive boy that she regularly "beat" on tests was the ringleader. During the entire time that she was in college as an undergraduate student, and during her first couple of years as a graduate student, Carolyn Gordon never met another woman mathematician. In addition, her mathematics courses consisted almost exclusively of men. Yet, she was not aware that this was an issue that bothered her until she attended a conference and went to an AWM (Association for Women in Mathematics) gathering during her third year in graduate school. When she walked into the roomful of women mathematicians at the AWM gathering, she was shocked by the experience, and realized that she had previously felt isolated without knowing what she had been missing.

Now, she manages to balance a successful research and teaching career at Dartmouth College with her family. She is married to a mathematician, David Webb, and they have a daughter. In addition, she is heavily involved in AWM. She feels that AWM has really helped women who have encountered barriers and active discrimination, and she has seen these benefits firsthand, through the experiences of friends and colleagues. She has mentored many women students and young faculty, including Dr. Sarah. She has seen the importance of role models, and has become one herself.

Mathematical Style

Carolyn Gordon has collaborated with many mathematicians. A search for her name on MathSciNet reveals 45 published papers with 14 different collaborators. She has other papers in the process of being written, including one with Dr. Sarah. Even though she is a geometer, she describes herself as being terrible at visualization and also as having a bad memory. While she is good with numbers, this skill does not help her in her research.

The following story also gives us insight into her mathematical style. The breakthrough that led to her research on hearing the shape of a drum occurred during Carolyn's talk at a conference when Wolpert, a member in the audience and future co-author, asked her a related question. Carolyn's husband said that the question "was like a cold shower. It really made us sit up and think about this…" Afterwards, Carolyn and her husband spent days making models and checking to see if they worked. Carolyn recalls "We got these huge (paper) castles. They took up our living room." The pieces were just starting to fall into place when their research was interrupted by a short trip by Carolyn to Germany. They completed the research with "a lot of transatlantic phone calls and twice-a-day faxes," Webb recalls.

In order to get the flashes of insight needed to do research, she finds that sometimes, after she has been working really hard and is stuck, she needs to step away from the problem and let her subconscious work. Sometimes, but not always, while she is partially occupied with something else, this is precisely when the new ideas will come to her. She describes the process of doing research like being in a maze. "Sometimes, when you are completely lost, you have taken a wrong turn, and you must back away and try a new direction. Other times, you will reach a door, find a way to open it, and then will find that you have made progress and entered a deeper, more significant part of the maze".

Mathematics

Much of what scientists know about our world comes from indirect observations. For example, X-rays, CAT scans, and other medical imaging techniques are indirect ways of seeing inside the body. Mathematicians and physicists are also concerned with indirect ways of observing the world around them. Carolyn Gordon's research on hearing the shape of a drum relates to indirect observations, which we will explore in this section.

Around the middle of the 19^{th} century, the French philosopher Comte speculated that knowledge of the chemical composition of stars would be forever beyond the reach of science. He was wrong and soon afterwards the related field of spectroscopy was created. Molecules in interstellar space are identified by their natural vibration frequencies, ie by the pitches at which a molecule naturally "rings". Given a set of vibration frequencies, an important research topic is to ask what can be inferred about the system's structure.

In 1966, the Polish-American mathematician Mark Kac asked a related question about whether one can always *hear* the shape of a drum. In other words, if you close your eyes and listen to differently shaped drums being played, Mark Kac wanted to know if you could distinguish the shape by the sound or vibration frequencies that you hear. A **mathematical drum** is not a standard musical instrument: It is any shape in the plane that has an interior and a boundary, such as a circle, a square, or a triangle. The interior vibrates with each strike while the boundary frame remains rigid. Imagine that we had a machine that could tell us the exact frequency of the sound of the drum vibrations. In 1911, Hermann Weyl proved that one can always hear the area of a mathematical drum which makes sense as the bigger the area of the drum, the lower the tone. Later, Minakshisundaram and Pleijel proved that one can always hear the length of the boundary, or the perimeter of the drum. It was thought that the sound of a drum might contain enough geometric information to specify its shape uniquely, and Mark Kac asked whether this was true.

The problem challenged researchers for nearly three decades. Finally, in the spring of 1991, Carolyn Gordon, her husband David Webb, and the audience member Scott Wolpert came up with the answer: No! One can sometimes, but not always hear the shape of a drum. They found two mathematical drums that have different shapes, (see Figure 2) but still had the same vibration frequencies, therefore making the same exact sound.

A mathematical proof does not need to be constructive. In fact, in this case, mathematicians would not have been satisfied with experiments to show that the drums sound the same because calculations and experiments cannot be exact and a very small frequency difference could escape experimental detection. Instead, Carolyn Gordon and her collaborators used mathematics to prove that her drums sounded the same without actually testing them in real-life. Later on, physicists created the drums and tested them in real life. They found that the drums sounded nearly the same with an error attributed to the experimental procedures. Carolyn Gordon's research on hearing the shape of a drum shows us there is not always just one conclusion that can be reached from a complete set of measurements.

Classroom Activities

The following classroom activities are designed to reinforce learning and engage the reader (see the separate sheets to turn back in). **The drums that Carolyn found were differently shaped, sounded the same and hence also had had the same area and perimeter.**

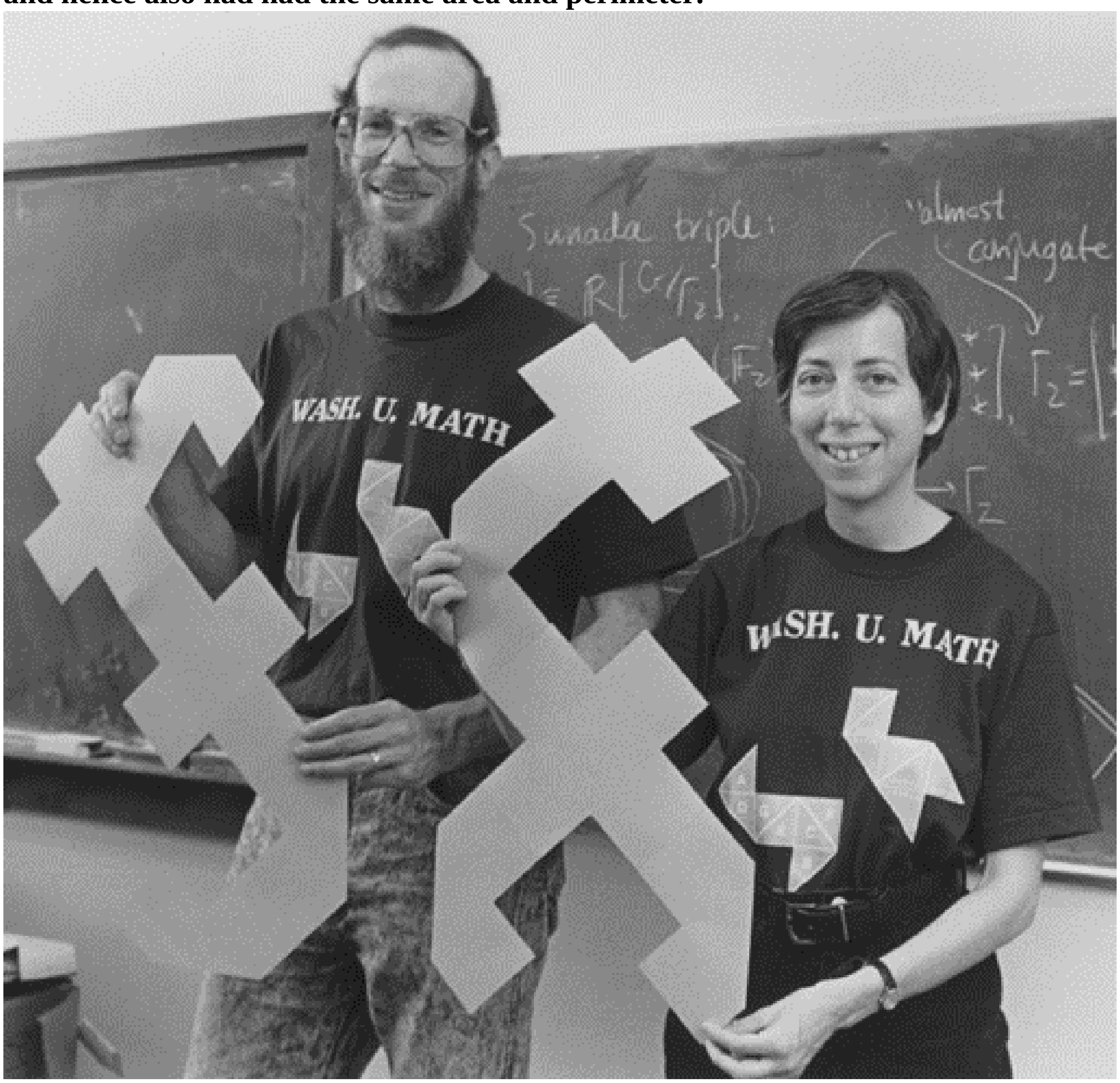

Figure 2. Carolyn Gordon and husband David Webb with their drums in 1991. Reprinted with permission of Washington University.

Continue by following the directions on the separate sheets.

References and Comments on How I Used Them

Cipra, Barry. (1992). *You can't always hear the shape of a drum*. Science, March 27, 1992, Volume 255, No. 5052, p. 1642 --?
This magazine article had an overview of the problem and a description of the reaction their reaction to Wolpert's question, their model building, and transatlantic work.

Cipra, Barry. (1997a). *You can't always hear the shape of a drum* [On-line]. Available: http://www.ams.org/new-in-math/hap-drum/hap-drum.html
This is a great website that I used to find an overview of the history and solution of the problem.

Gordon, Carolyn and David Webb. *You Can't Hear the Shape of a Drum*. American Scientist, January-February, 1996, Volume 84, No. 1, p. 46 --?.
This magazine article had a great summary of the solution of the problem, and some information about the authors. They recently received The Chauvenet Prize for writing this article, given for an outstanding expository article on a mathematical topic by a member of the Mathematical Association of America.

Mathscinet search on Carolyn Gordon. (2001) [On-line]. Available: http://www.ams.org/mathscinet
I used this site to find her published papers and collaborators.

Personal communication with Carolyn Gordon (2001).

Peterson, Ivars. (1997a). *Ivars Peterson's MathLand: Drums that sound alike* [On-line]. Available: http://www.maa.org/mathland/mathland_4_14.html
I used this site to find the pictures in Figure 3 and 4, and it also contained information about the physicists who made the drums and performed experiments to show that they sounded the same.

Weintraub, Steven. (1997). *What's new in mathematics – June 1997 cover* [On-line]. Available: http://www.ams.org/new-in-math/cover/199706.html
This website contains the pictures of Carolyn Gordon and David Webb holding their drums. It also contains links to an animated picture of the frequency and waves when the drums are struck.

I could not find information in books about Carolyn Gordon.

NAME____

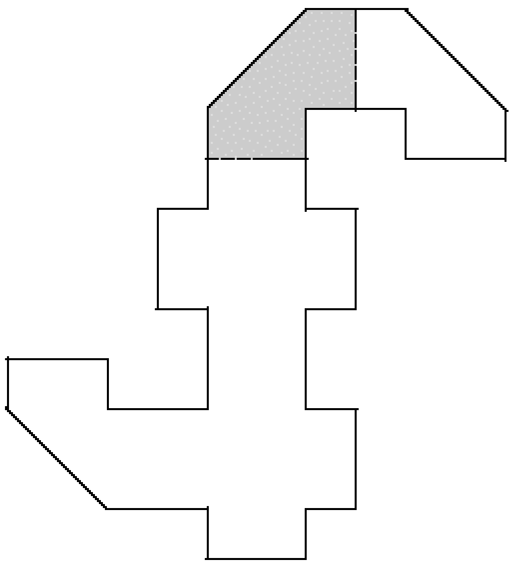

Figure 3: David Webb's Drum in Fig 2

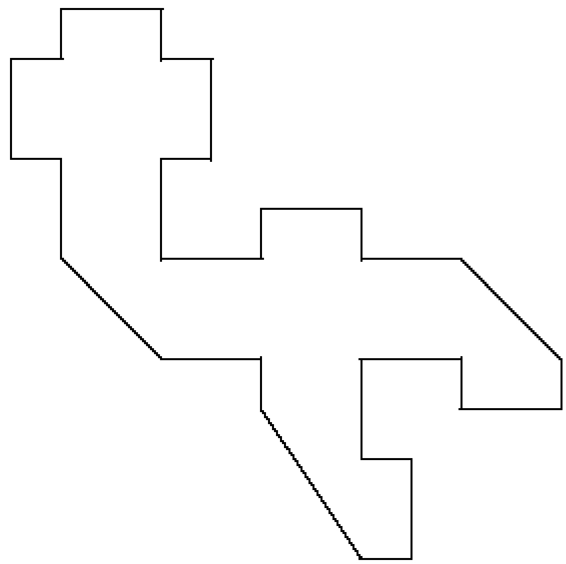

Figure 4: Carolyn Gordon's Drum in Fig 2

1) Cut along the boundaries of the soundalike drums in Figures 4a and 4b (pull off the next page) **which are duplicate copies of Figure 4.** If the drums in Figures 3 and 4 have the same shape, then you will be able to place one on top of the other. Place Figure 4a on top of Figure 3 above, and try to move figure 4a (via rotating, translating or flipping it around) so that it matches Figure 3. **<u>Question</u>:** Do these figures have the same shape?

2) Take Figure 4b (the one with the dashed lines marked on it) and <u>cut</u> this drum along the dashes. Notice that you will have 5 pieces total. Two pieces will be red crosses of the same size, and the other three pieces will be halves of a cross (split along the diagonal of the square in the middle of a cross) that resemble the shaded part of Figure 3. Try to fit these cut pieces onto Figure 3. Notice that you won't be able to do so. In fact, no matter how you might cut Figure 4b (don't try this now), you won't be able to fit it onto Figure 3. This contradicts the fact that they must have the same area in order to sound the same. Let's try and resolve the apparent conflict by investigating the accuracy of the models represented. **Identify which piece** does not fit properly onto Figure 3 above. Look at Figure 4 above and **<u>put a star</u> on this piece on Figure 4 and <u>also put a star</u> on this piece on Figure 4a**, which is the drum that you cut out along the boundary but left in one piece.

3) Take Figure 4a and compare it to the drum Carolyn is actually holding in Figure 2 back on the worksheet. Compare the piece that you starred with the same piece on Carolyn's drum. Notice that the vertical edge next to the part that Carolyn is holding is the same length as the vertical edge opposite it, but that this is not true of your starred piece and its opposite edge. The drum that Carolyn is holding is drawn correctly, but the starred piece on Figures 4 and 4a was not correctly drawn to scale and it is this error in scaling that causes the contradiction in 2). In Figure 4 above, try to **<u>fix the problem piece and edge so that it is drawn to scale by adding to Figure 4</u>** to show how you would have drawn the piece. Take one of the other similar but correctly scaled pieces and place it on top of the problem piece in order to help you fix it.

The point of this exercise is to have you engage the models instead of just hearing about them (no pun intended). There are dangers in relying on models since it is difficult to create physical models representing abstract figures with precisely determined sides. These models were found on a webpage that discussed Carolyn Gordon's solution of the problem and the physicists' work that followed (see my comments in the references). The drums in the picture of Carolyn Gordon and David Webb are drawn to scale, have the same area and perimeter, but are shaped differently. If you close your eyes and listen to them being played, you cannot tell that they have different shapes, since they sound exactly the same. Carolyn Gordon's research on hearing the shape of a drum shows us that a mathematical proof does not need to be constructive and that there is not always just one conclusion that can be reached from a complete set of measurements.

Carolyn Gordon's research answer's Kac's question, but it raises many new issues. For example, now that we know that one cannot hear every property of a drum, what properties besides its area and perimeter really are audible? In addition, we know a great deal more about the forces that produce the vibrations of sound than about those that produce the vibrations of light. The problem that spectroscopy hopes to solve is whether one can recover the chemical composition of stellar atmospheres by using vibration "fingerprints" in order to identify them. Carolyn's research is a small step in this direction.

Specific Constructive Suggestions for Improvement

(Give very specific suggestions to help improve this worksheet. For example, if you find awkward wording in a sentence, then specify which sentence and try to improve the wording yourself.)

Positive Feedback

(Anytime one gives constructive suggestions, it is also a good idea to say something positive!)

Make sure that your name is on the other side, that you have answered question 1, that Figure 4 has a star on the correct piece as designated in question 2, that you have fixed the problem piece as directed in question 3, and that you have filled in this side of the sheet before you turn this in.

Cut along the boundary of each figure below:

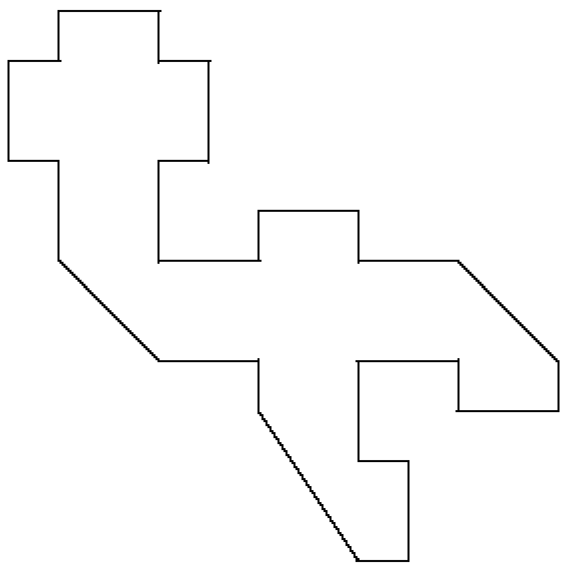

Figure 4a: Carolyn Gordon's Drum

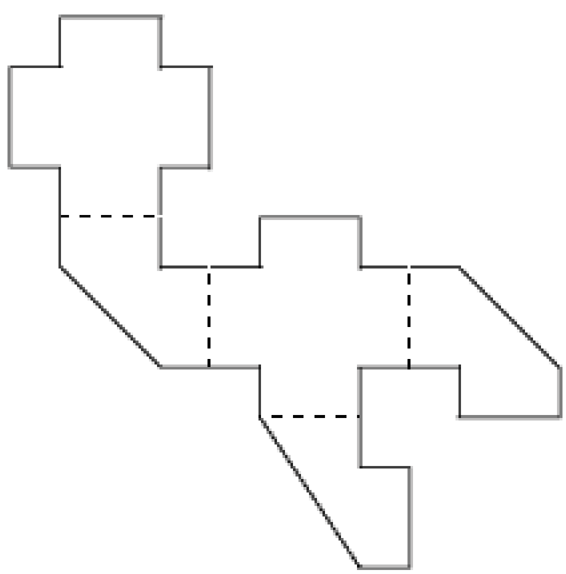

Figure 4b: Carolyn Gordon's Drum with dashes added