COFFEE COOLING ON A TI-CBL UNIT AND IN MAPLE

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Abstract

At the end of a chapter on Newton's Law of Cooling, Nagle and Saff's *Fundamentals of Differential Equations and Boundary Value Problems* [NS, pg 107] poses the following problem:

> Two friends sit down to talk and enjoy a cup of coffee. When the coffee is served, the impatient friend immediately adds a teaspoon of cream to his coffee. The relaxed friend waits 5 minutes before adding a teaspoon of cream (which has been kept at a constant temperature). The two now begin to drink their coffee. Who has the hotter coffee? Assume that the cream is cooler than the air and use Newton's Law of Cooling.

We will demonstrate a classroom experiment of this problem using a TI-CBL™ unit, hand-held technology that comes with temperature and other probes. We will explain how to import the data into Maple™, and look at an interactive Maple demo created to complement the experiment in which students must answer questions to progress. We will then discuss problems that arose with using the CBL Unit in this experiment, such as the fact that in the first experimental run, the impatient friend had cooler coffee, while in the second experimental run the relaxed friend had cooler coffee.

Using results obtained from the simplified experiment, we will look at an improved attempt at a classroom experiment.

For Maple Demos and other information, see http://www.mathsci.appstate.edu/∼sjg/ICTCM13/

Introduction

In a differential equations class, coffee cooling is a natural topic which introduces physical modeling. It is now possible to easily conduct a classroom experiment on cooling and obtain data using technology which is becoming popular in high school and college classrooms. The Texas Instruments CBL Unit is a hand-held data logger that comes with probes to measure voltage, distance, and temperature. Over 40 other sensors are available for a CBL. A TI-CBL links to TI calculators. Casio has recently introduced similar technology with the EA100 Data Analysis System.

After discussing a cooling experiment using a CBL, we will highlight some of the problems that occurred in the classroom.

The First Experiment

Nagle & Saff describe a coffee cooling thought experiment in which cream is added immediately to one cup of coffee and later to a second cup. Students are asked the question, "Which coffee is cooler after the second cream is added?" This experiment makes a very lively classroom activity. (See also [RV] for a physics approach to the experiment.)

Student teams are broken into a Coffee Pourer, a Stopwatch Timer, a Data Recorder, and two Cream Pourers. The experimental procedure for Black Coffee Cooling is:

1. Set up a CBL unit and calculators to measure every 4 seconds for 99 readings giving a total of 396 seconds.
2. When the Timer says, "go," the following occur: The coffee pourer pours the coffee into the mug. The stopwatch starts counting to 5.5 minutes. The CBL unit is triggered to collect data.
3. The temperature watcher records the highest temperature achieved before cooling.
4. When the stopwatch timer beeps at 5.5 minutes, the temperature watcher records the temperature and the cream pourers each pour a creamer into the coffee.
5. The temperature watcher records the final temperature.

The experimental procedure for Cafe Au Lait Coffee Cooling is:

1. Set up CBL unit and calculators to measure every 4 seconds for 99 readings for a total of 396 seconds.
2. When the Timer says, "go," the following occur: The coffee pourer pours the coffee into the mug. The stopwatch starts counting to 30 seconds. The CBL unit starts measuring data.
3. When the stopwatch timer beeps at 30 seconds, the temperature watcher records the temperature and the cream pourers each pour a creamer into the coffee.
4. The temperature watcher records the final temperature.

The collected data is transfered to calculators and then uploaded to a computer via a Graph Link in order to be analyzed with Maple. Students work through a prepared Maple Demo ([SG]) that guides their progress. They must answer questions as they work through the worksheet.

Simplified Experiment

Repeated trials of this experiment may yield contradictory results. There are many sources of possible error in the procedure, including, for example, varying responses of different temperature probes, differing evaporation rates for black and creamed coffee ([RV]), or temperature differences due to convection. One way to reduce error is to simplify the experiment. We chose to eliminate the cream and study black coffee cooling.

Analysis

Once we have chosen a model and collected data, estimating the model's parameters becomes our next task. Rewrite Newton's Law of Cooling as

$$\begin{aligned}\frac{dT}{dt} &= k \cdot (M - T(t)) \\ &= a + bT\end{aligned}$$

where $k = -b$ and $M = -a/b$. We will use a divided difference to approximate the derivative and arrive at

$$\frac{\Delta T}{\Delta t} = a + bT$$

Let $h = \Delta t$, which is constant throughout the interval, then we have several choices of approximators:

$$\begin{aligned}\frac{\Delta T}{\Delta t} &\approx \frac{T_i - T_{i-1}}{h} \quad \textit{Forward Difference}, O(h, f'') \\ &\approx \frac{T_{i+1} - T_{i-1}}{2h} \quad \textit{Three Point Difference}, O(h^2, f''')\end{aligned}$$

We can use a concavity argument to convince differential equation students that a three point formula will be superior to a forward or backward difference. Students can also estimate the error by differentiating the differential equation. In a higher level course, we could consider using a five point formula.

$$\frac{\Delta T}{\Delta t} \approx \frac{T_{i+2} - 8T_{i+1} + 8T_{i-1} - T_{i-2}}{12h} \quad \textit{Five Point Difference}, O(h^4, f^{(5)})$$

Once we have chosen to use the three point approximation to the derivative dT/dt, we consider our data in the form

$$[T_i, \Delta T_i/\Delta t]$$

and perform a linear regression to determine the coefficients a and b, and thus, k and M.

If we have measured the ambient temperature M separately, an alternate method is possible. We can use the mean of relative divided differences of the data to estimate the single remaining parameter k, since, from Newton's Law of Cooling, we have

$$k = \frac{\Delta T_i}{\Delta t} \cdot \frac{1}{M - T_i}$$

With the values of M and k in hand, symbolically solving Newton's Law gives the model

$$T = M + (T_0 - M)\, e^{-kt}$$

A different approach, more often used in physics circles, is to solve the differential equation symbolically, and then estimate the parameters using this different form. The basic concept of transforming the data in order to more easily fit the model is still the theme.

Rewrite the solution of Newton's Law relabeling $T - M = \tilde{T}$,

$$\tilde{T} = \tilde{T}_0\, e^{-kt}$$

Now apply logarithms to the equation.

$$\log(\tilde{T}) = \log(\tilde{T_0}) - kt$$

(Physicists usually write $k = 1/\tau$.) This form of the solution leads us to use a linear regression on the pairs $[t, \log(\tilde{T_i})]$ in order to determine k and $\log(\tilde{T_0})$. A good student project for an advanced group is to ask for an analysis of the difference between these parameter estimating methods.

Conclusion

As one might expect in any experiment, we encountered error and simplified the procedure to reduce the problems. Parameter estimation as we outlined above is reasonably successful (for example, see [SG]).

Return to our original problem. It is not difficult to show that, for cream that is cold (below ambient temperature), we should add the cream early to have the hottest coffee at 5.5 minutes, while for cream that is hot (above ambient temperature), we should add the cream late to have the hottest coffee.

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References

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